URTHER SECOND ORDER ROTATABLE DESIGNS

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1. Introduction

Since the introduction of Second Order Rotatable Design (SORD) by Box and Hunter (1957), many authors have presented different techniques for the construction of these designs. However, very few designs with factors each at six levels are available in literature. Das (1961) has given one such design in two dimensions. Nigam and Dey (1970) have also reported a six-level design in three dimensions. Using the properties of Partially Balanced Arrays, Gupta and Dey (1973) have recently obtained a general class of 6-level SORD. In the present paper, a new class of 6-level SORD is presented.

2. Method of Construction

Let there exist a Balanced Incomplete Block (BIB) design D, with parameters v, b, r, k, λ and let $N=(n_{ij})$ be its incidence matrix, where

$$n_{ij}=1, \text{ if the } j\text{-th treatment occurs in the } i\text{-th block}$$

$$=0, \text{ otherwise.}$$

In D, replace the zero by α and unity by β to obtain an array D_1 of b rows and v columns involving α and β. Next in D, replace the zero by η and unity by β to obtain another $b \times v$ array D_2, involving β and η. Let the array with $2b$ rows and v columns, comprising of D_1 and D_2 be called A^*. Now, we 'multiply' each of these $2b$ combinations of A^* with those of a 2^p factorial with levels ± 1, where, 2^p is the smallest fraction of 2^v without confounding any interaction of third order or less and the term "multiplication" is in the sense of Das and Narasimham (1962). This process of multiplication leads us to $b.2^{p+1}$ design points, each of v-dimension. These points, evidently satisfy the following :

(A) $$\sum_u \left\{ \prod_{i=1}^{v} x_{iu}^{\alpha_i} \right\} = 0, \text{ if any } \alpha_i \text{ is odd, } \alpha_i = 0, 1, 2 \text{ or } 3 \text{ and}$$

$$\sum_i \alpha_i \leqslant 4.$$

(B) $$\sum_u x_{iu}^2 = (b-r)\alpha^2 + 2r\beta^2 + (b-r)\eta^2,$$

$$\sum_u x_{iu}^4 = (b-r)\alpha^4 + 2r\beta^4 + (b-r)\eta^4,$$

$$i = 1, 2, \ldots, v.$$

(C) $$\sum_u x_{iu}^2 \, x_{ju}^2 = (b-2r+\lambda)\alpha^4 + 2(r-\lambda)\alpha^2\beta^2 + 2\lambda\beta^4$$

$$+ (b-2r+\lambda)\eta^4 + 2(r-\lambda)\beta^2\eta^2,$$

$$i \neq j,$$

$$i, j = 1, 2, \ldots, v.$$

In the above expressions, x_{iu} denotes the i-th coordinate in the u-th design point and summation is over the design points.

The above conditions are among the requirements of a SORD. Further, the design points must also satisfy the following two conditions for a SORD :

(D) $$3\sum_u x^2_{iu} \, x^2_{ju} = \sum_u x^4_{iu},$$

(E) $$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{(v+2)} \quad \text{where} \quad \lambda_2 = \sum_u x^2_{iu}/N,$$

and $$\lambda_4 = \sum_u x^2_{iu} \, x^2_{ju}/N.$$

The $b.2^{p+1}$ points obtained above satisfy the condition (E), as not all these points are equidistant from the origin.

Applying condition (D) on the design points, we have the following equation after simplification :

$$(2b-5r+3\lambda)\alpha^4-2(r-3\lambda)\beta^4+(2b-5r+3\lambda)\eta^4$$
$$+6(r-\lambda)\alpha^2\beta^2+6(r-\lambda)\beta^2\eta^2=0 \quad ...(2.1)$$

Substituting u for α^2/η^2 and w for β^2/η^2 in (2·1) we obtain

$$(2b-5r+3\lambda)\, u^2-2(r-3\lambda)w^2+6(r-\lambda)uw+6(r-\lambda)w$$
$$+(2b-5r+3\lambda)=0. \quad ...(2.2)$$

In order that the degin exists, (2·2) must admit positive solutions for u and w. Since (2·2) involves two unknowns u and w, we fix one of them arbitrarily, so that the solution for the other is positive. Now, we take up the three cases, *viz.*, (*i*) $r=3\lambda$ (*ii*) $r>3\lambda$ and (*iii*) $r<3\lambda$ separately.

Case 1. $r=3\lambda$.

In this case, (2·2) reduces to

$$(b-2r)u^2+2ruw+2rw+(b-2r)=0 \quad ...(2{\cdot}3)$$

Dey and Das (1970) have shown that for any BIB design $b\geqslant 3(r-\lambda)$. Thus, when $r=3\lambda$, $b\geqslant 2r$. Using this fact, one can easily see that (2·3) cannot have a positive solution for w for any arbitrary value of $u>0$. This shows that the present method of construction does not give a 6-level SORD if BIB designs satisfying $r=3\lambda$ are used.

Case 2. $r>3\lambda$.

Fixing arbitrarily $w=2$, we can reduce (2·2) to

$$(2b-5r+3\lambda)u^2+12(r-\lambda)u+(2b-r+15\lambda)=0. \quad ...(2{\cdot}4)$$

Solving the above quadratic in u, we have

$$u=\frac{1}{\Delta}[-12(r-\lambda)\pm\sqrt{\{144(r-\lambda)^2-4(2b-5r+3\lambda)}$$
$$(2b-r+15\lambda)\}], \quad ...(2.5)$$

where $\Delta=2(2b-5r+3\lambda)$.

From (2·5) it is clear that if $2b-5r+3\lambda<0$ a positive solution of u always exists, for, $2b-r+15\lambda$ is always positive. Now, we prove the following

Lemma 2.1. If in a BIB design, $r>3\lambda$ then

$$2b-5r+3\lambda \geqslant 0.$$

Proof : Let $S=2b-5r+3\lambda$.

Then $$S=\{b-3(r-\lambda)\}+(b-2r) \qquad ...(2\cdot6)$$

Now since $b \geqslant 3(r-\lambda)$ and $r>3\lambda$, it follows that $b \geqslant 2r$. Thus, $S \geqslant 0$.

From the above lemma, it is clear that we cannot use BIB designs satisfying $r>3\lambda$ for constructing 6-level SORD.

Case 3. $r<3\lambda$.

In this case also, if $2b-5r+3\lambda<0$, we get a positive solution for u. We now prove

Lemma 2.2. For a BIB design, $2b-5r+3\lambda<0$

if $$2v<3k+2.$$

Proof. Let $$T=2b-5r+3\lambda$$

$$=2b-2r-3(r-\lambda).$$

Also,

$$r-\lambda=r-\frac{r(k-1)}{v-1}=r(v-k)/(v-1),$$

Thus,

$$T=r(v-k)(2v-3k-2)/\{(v-1)k\}.$$

Hence the lemma.

Thus, if we have a BIB design satisfying (*i*) $r<3\lambda$ and (*ii*) $2v<3k+2$, we get a 6-level SORD. However, in order that the six levels are distinct we must show that the value of u obtained is not equal to unity.

We have, under the condition that $2b-5r+3\lambda<0$, the positive solution of u as

$$u=\frac{-12(r-\lambda)-\sqrt{[144(r-\lambda)^2-4(2b-5r+3\lambda)(2b-r+15\lambda)]}}{2(2b-5r+3\lambda)}$$

Let $$f(u)=u^2-1,$$

Then,

$$f(u)=\frac{144(r-\lambda)^2+A+24(r-\lambda)\sqrt{A}}{4(2b-5r+3\lambda)^2}-1,$$

$$A=144(r-\lambda)^2-4(2b-5r+3\lambda)(2b-r+15\lambda).$$

Therefore,

$$f(u)=\frac{288(r-\lambda)^2-4(2b-5r+3\lambda)(4b-6r+18\lambda)+24(r-\lambda)\sqrt{A}}{4(2b-5r+3\lambda)^2}$$

Thus, if $4b-6r+18\lambda>0, f(u)>0$

Let $M=4b-6r+18\lambda$

$$=2[b+6\lambda+\{b-3r+3\lambda\}]$$

Since $b\geqslant 3(r-\lambda)$, $M>0$

Thus, $f(u)=u^2-1>0.$

An Example. We construct a 3-factor SORD. Let us consider the BIB design with parameters $v=b=3$, $r=k=2$, $\lambda=1$. This design satisfies both the requirements, *viz.*, $r<3\lambda$ and $2v<(3k+2)$. The design points of the SORD are shown below :

β	β	α	β	α	β	α	β	β	β	β	η	β	η	β	η	β	β
β	β	$-\alpha$	β	α	$-\beta$	α	β	$-\beta$	β	β	$-\eta$	β	η	$-\beta$	η	β	$-\beta$
β	$-\beta$	α	β	$-\alpha$	β	α	$-\beta$	β	β	$-\beta$	η	β	$-\eta$	β	η	$-\beta$	β
β	$-\beta$	$-\alpha$	β	$-\alpha$	$-\beta$	α	$-\beta$	$-\beta$	β	$-\beta$	$-\eta$	β	$-\eta$	$-\beta$	η	$-\beta$	$-\beta$
$-\beta$	β	α	$-\beta$	α	β	$-\alpha$	β	β	$-\beta$	β	η	$-\beta$	η	$-\beta$	$-\eta$	β	β
$-\beta$	β	$-\alpha$	$-\beta$	α	$-\beta$	$-\alpha$	β	$-\beta$	$-\beta$	β	$-\eta$	$-\beta$	η	$-\beta$	$-\eta$	β	$-\beta$
$-\beta$	$-\beta$	α	$-\beta$	$-\alpha$	β	$-\alpha$	$-\beta$	β	$-\beta$	$-\beta$	η	$-\beta$	$-\eta$	β	$-\eta$	$-\beta$	β
$-\beta$	$-\beta$	$-\alpha$	$-\beta$	$-\alpha$	$-\beta$	$-\alpha$	$-\beta$	$-\beta$	$-\beta$	$-\beta$	$-\eta$	$-\beta$	$-\eta$	$-\beta$	$-\eta$	$-\beta$	$-\beta$

In this case, $w=2$, $u=13{\cdot}416$.

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